Stackelberg games, optimal pricing and application to electricity markets

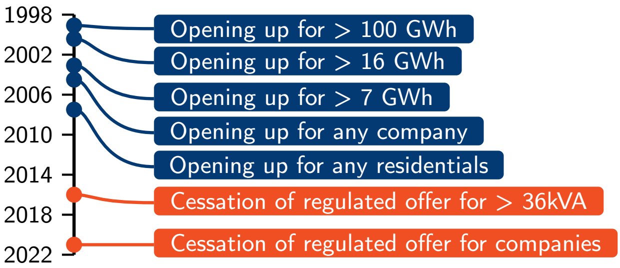
Supervised by Stéphane Gaubert, Wim van Ackooij and Clémence Alasseur

October 24, 2023

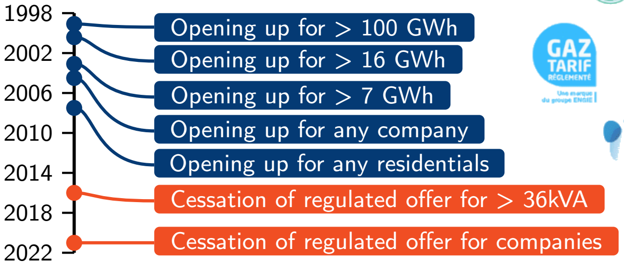
Quentin Jacquet

CONTEXT AND MOTIVATIONS

A competitive market

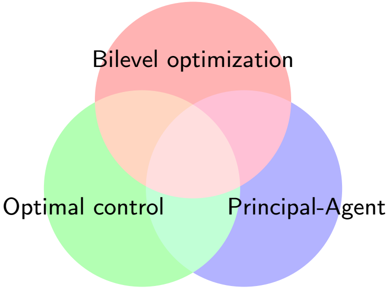


A competitive market



Source: Hello Watt

Roadmap

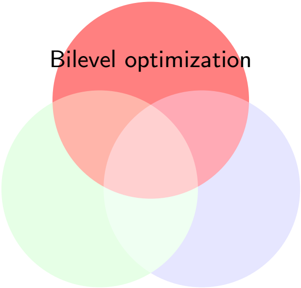


- **Chapter 4: Bilevel optimization**
  A retailer optimizes prices of existing offers *by taking into account* the rational behavior of customers (choice of the optimal tariff).

- **Chapter 5: Optimal control**
  A retailer finds an optimal *policy* to maximize a gain on a period considering the *dynamics* of the population (shift from one offer to another).

- **Chapter 6: Principal-Agent model**
  A retailer designs an optimal *contract* (function depending on the consumption level) to a continuum of agents.

Roadmap



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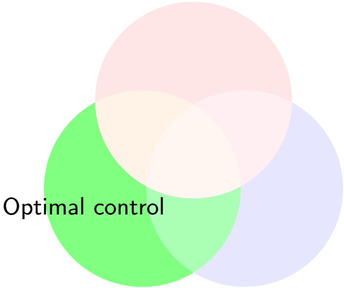

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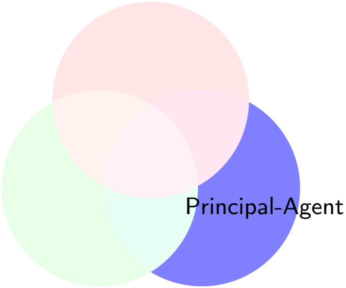

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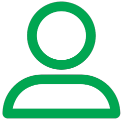
Stackelberg games[1]

Leader
Controller
Principal

$$\max_{x \in \mathcal{X}, y^*} \quad F(x, y^*)$$
$$\text{s.t.} \quad y^* \in \Psi(x) = \operatorname*{arg\,min}_{y \in \mathcal{Y},\, g(x,y) \leq 0} \; f(x, y) \; .$$



Follower
Environment
Agent

[1] H. von Stackelberg. "Theory of the Market Economy" (1952)

Stackelberg games[1]

Upper decision / action / control x

Leader
Controller
Principal

$$\max_{x \in \mathcal{X}, y^*} \quad F(x, y^*)$$

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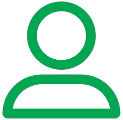

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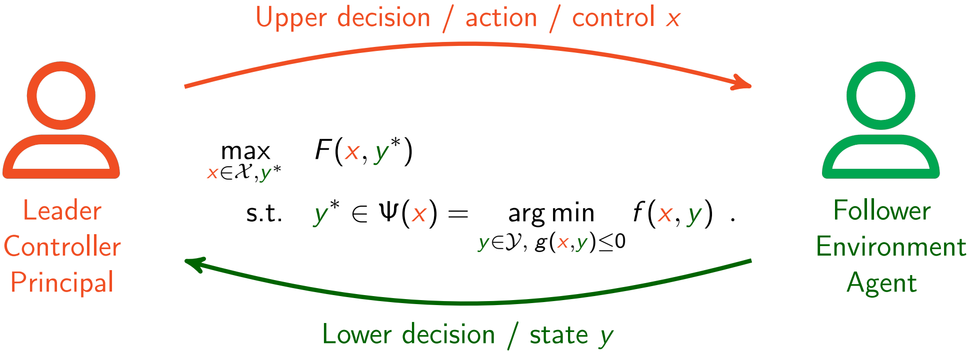
Stackelberg games[1]



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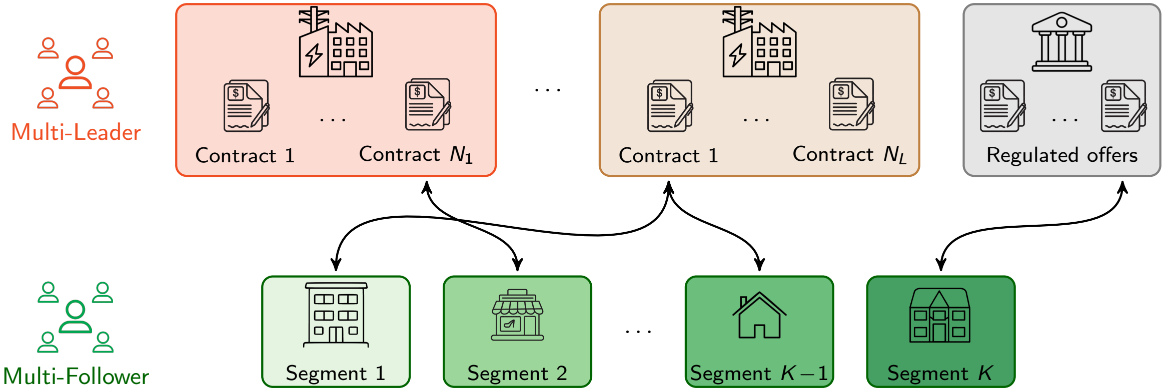
[1] H. von Stackelberg. "Theory of the Market Economy" (1952)

STUDY OF CUSTOMERS BEHAVIOR IN BILEVEL PRICING PROBLEMS

Q. J., W. van Ackooij, C. Alasseur, and S. Gaubert. "Quadratic regularization of bilevel pricing problems and application to electricity retail markets". In: *European Journal of Operational Research* (May 2023)

Actors involved in the market



$\rightsquigarrow$ Nash equilibrium at upper level[1]

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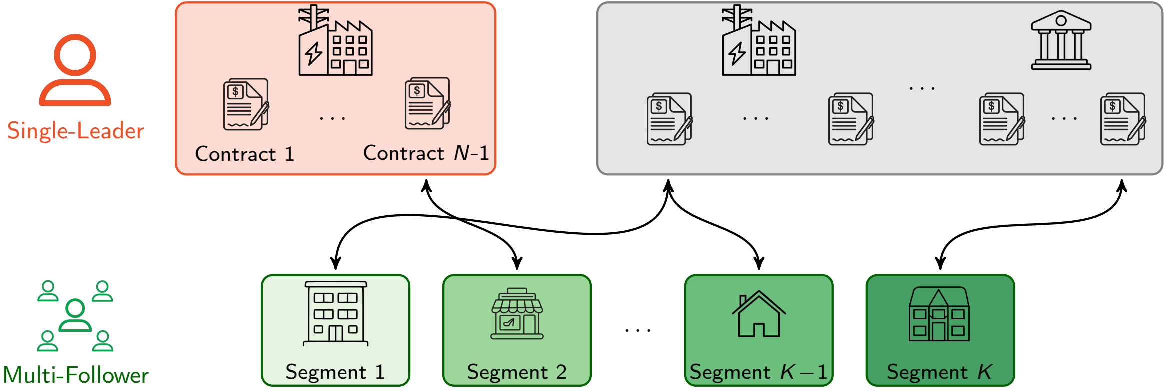
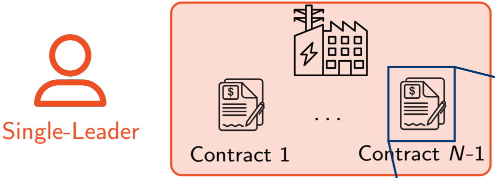
[1] S. Leyffer and T. Munson. "Solving multi-leader–common-follower games". In: *Optimization Methods and Software* 25.4 (2010), pp. 601–623

Actors involved in the market



⇝ ~~*Nash equilibrium at upper level*~~ → static competition

Actors involved in the market



Contract structure:

2 classical versions (Baseload, Peak/Off-peak)

	Baseload	Peak/Off-peak
Variable portion (€/kWh)	unique price	peak price
		off-peak price
Fixed portion (€)	power	power

D attributes

(Envy-free) Product Pricing problem [1]

Notation:

- $[K] := \{1 \dots K\}$ customers segments,
- $[N]$ contracts (the N-th is the alternative),

Variables:

- $x_n \in \mathbb{R}^D$ price vector for contract n,
- $\mu_{kn} = \begin{cases} 1 \text{ if segment } k \text{ chooses } n, \\ 0 \text{ otherwise.} \end{cases}$

Data:

- C_{kn} cost to supply k if he chooses n,
- R_{kn} reservation price of k for contract n,
- $E_{kn} \in \mathbb{R}_+^D$ fixed consumption of k.

Unitary profit and utility:

$$\theta_{kn}(x) := \underbrace{\langle E_{kn}, x_n \rangle_D}_{\text{electricity invoice}} - \underbrace{C_{kn}}_{\text{cost}} \quad , \quad \theta_{kN} = 0$$

$$U_{kn}(x) := \underbrace{R_{kn}}_{\text{reservation price}} - \underbrace{\langle E_{kn}, x_n \rangle_D}_{\text{electricity invoice}} \quad , \quad U_{kN} = 0$$

Profit-maximization problem:

$$\begin{cases} \boxed{\max_{x \in \mathcal{X}, \mu^*} \; J(x) := \sum_{k \in [K]} \rho_k \langle \theta_k(x), \mu_k^* \rangle_N} \rightarrow \text{leader pb} \\ \text{s. t.} \quad \boxed{\mu_k^* \in \arg\max_{\mu \in \Delta_N} \langle U_k(x), \mu_k \rangle_N} \rightarrow \text{follower pb} \end{cases}$$

[1] M. Labbé, P. Marcotte, and G. Savard. "A bilevel model of taxation and its application to optimal highway pricing". In: *Management science* 44 (1998), pp. 1608–1622

Price complex and instability

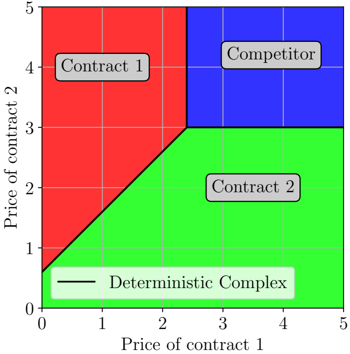


Figure: Follower response[1], $(K = 1,\ N = 3)$

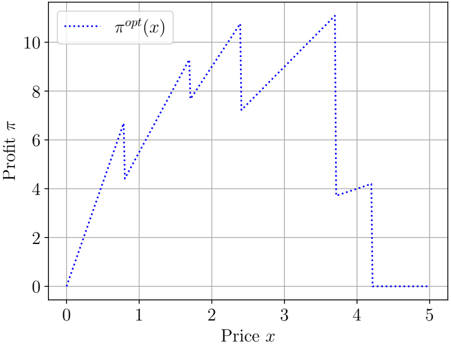


Figure: Profit function, $(K = 5,\ N = 2)$

[1] E. Baldwin and P. Klemperer. "Understanding preferences:"demand types", and the existence of equilibrium with indivisibilities". In: *Econometrica* 87.3 (2019), pp. 867–932

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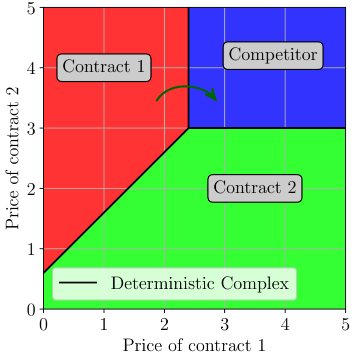
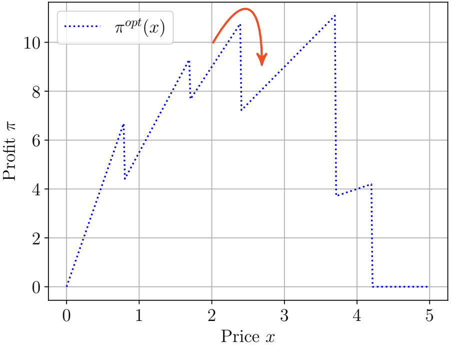


Figure: Follower response[1], ($K = 1$, $N = 3$)



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Mixed Multinomial Logit model (MMNL)

$$\begin{cases} \max_{x\in\mathcal{X},\mu^*} \sum_{k\in[K]} \rho_k \langle \theta_k(x), \mu_k^* \rangle_N \\ \text{s. t.} \quad \mu_k^* \in \arg\min_{\mu\in\Delta_N} \left\{ \begin{array}{l} -\langle U_k(x), \mu_k \rangle_N \\ +\frac{1}{\beta} \langle \log(\mu_k), \mu_k \rangle_N \end{array} \right\} \end{cases}$$

$$\rightsquigarrow \mu_{kn}^*(x) = e^{\beta U_{kn}(x)} / \sum_{l\in[N]} e^{\beta U_{kl}(x)}$$

$\Rightarrow \mu_k^* \in \operatorname{Int} \Delta_N$, no polyhedral complex

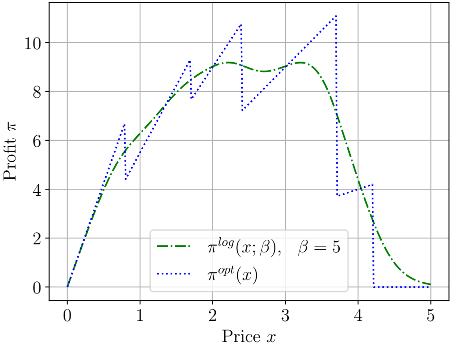

Figure: Logit regularization[1] ($K = 5, N = 2$)

[1] H. Li, S. Webster, N. Mason, and K. Kempf. "Product-Line Pricing Under Discrete Mixed Multinomial Logit Demand". In: *Manufacturing and Service Operations Management* 21 (2019), pp. 14–28
F. Gilbert, P. Marcotte, and G. Savard. "A Numerical Study of the Logit Network Pricing Problem". In: *Transportation Science* 49 (Jan. 2015), p. 150105061815001

Literature review

	Customers' response	Resolution
[Gur+05]	Deterministic	Complexity results
[STM11]	Deterministic	MILP + heuristics
[Fer+16]	Deterministic	MILP + valid cuts
[Eyt18]	Deterministic	Tropical methods
[BK19]	Deterministic	Tropical methods
[STH07]	Probabilistic	MILP
[GMS15]	Deterministic MMNL	Nonlinear optimization
[LH11]	MNL	Convex reformulation
[Li+19]	MMNL	Heuristics
[Hoh20]	MMNL	Nonlinear optimization
This work	**Quadratic**	MIQP + pivoting heuristics

Our approach: Quadratic regularization (1)

$$\begin{cases} \max\limits_{x \in \mathcal{X}, \mu^*} \sum\limits_{k \in [K]} \rho_k \langle \theta_k(x), \mu_k^* \rangle_N \\ \text{s. t.} \quad \mu_k^* \in \arg\min\limits_{\mu \in \Delta_N} \begin{Bmatrix} -\langle U_k(x), \mu_k \rangle_N \\ + \frac{1}{\beta} \langle \log(\mu_k), \mu_k \rangle_N \end{Bmatrix} \end{cases}$$

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$$\rightsquigarrow \mu_k^*(x) = \text{Proj}_{\Delta_N}\left(\frac{\beta}{2}(U_k(x))\right)$$

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$$\rightsquigarrow \mu_{kn}^*(x) = e^{\beta U_{kn}(x)} / \sum_{l \in [N]} e^{\beta U_{kl}(x)}$$

+ Probabilistic behavior ($\mu_k^* \in \operatorname{Int} \Delta_N$)
+ Explicit lower response
- No combinatorial structure (non-convex NLP)

$$\begin{cases} \max\limits_{x \in \mathcal{X}, \mu} \sum\limits_{k \in [K]} \rho_k \langle \theta_k(x), \mu_k^* \rangle_N \\ \text{s. t.} \quad \mu_k^* \in \arg\min\limits_{\mu \in \Delta_N} \left\{ \begin{aligned} & - \langle U_k(x), \mu_k \rangle_N \\ & + \frac{1}{\beta} \langle \mu_k - 1, \mu_k \rangle_N \end{aligned} \right\} \end{cases}$$

$$\rightsquigarrow \mu_k^*(x) = \operatorname{Proj}_{\Delta_N} \left(\frac{\beta}{2} (U_k(x)) \right)$$

+ Probabilistic behavior ($\mu_k^* \in \Delta_N$)
+ Fast projection algorithms[1]
+ Combinatorial structure (polyhedral complex)

[1] L. Condat. "Fast Projection onto the Simplex and the l1 Ball". In: *Mathematical Programming, Series A* 158.1 (July 2016), pp. 575–585

Our approach: Quadratic regularization (2)

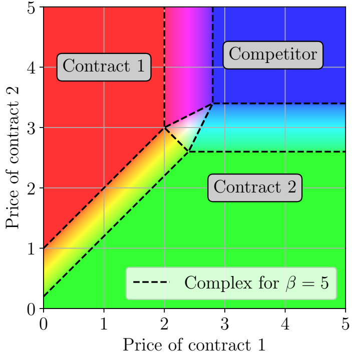


Figure: Follower response, ($K = 1$, $N = 3$)

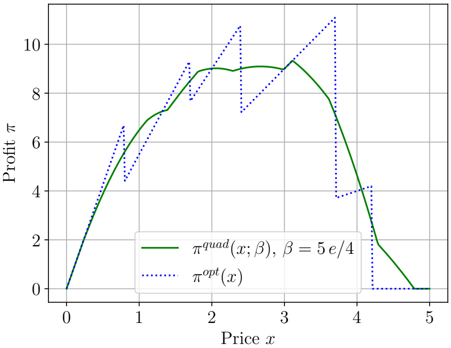


Figure: Profit function, ($K = 5$, $N = 2$)

Theorem:

The decision of the customers remains a polyhedral complex. Moreover, the profit is continuous and *concave* on each cell of the polyhedral complex.

Customers' response as a polyhedral complex

Envy-free PPP is APX-Hard[1]

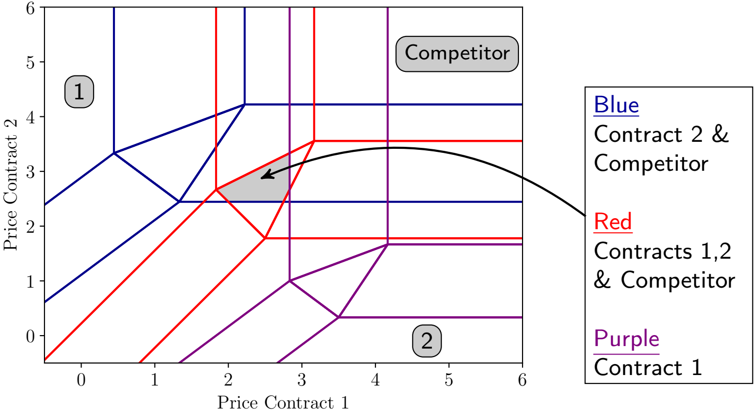


Figure: Polyhedral complex with $K = 3$ segments and $N = 3$ contracts

[1] V. Guruswami, J. D. Hartline, A. R. Karlin, D. Kempe, C. Kenyon, and F. McSherry. "On profit-maximizing envy-free pricing.". In: *SODA*. vol. 5. 2005, pp. 1164–1173

Design of a pivoting heuristic – On an example

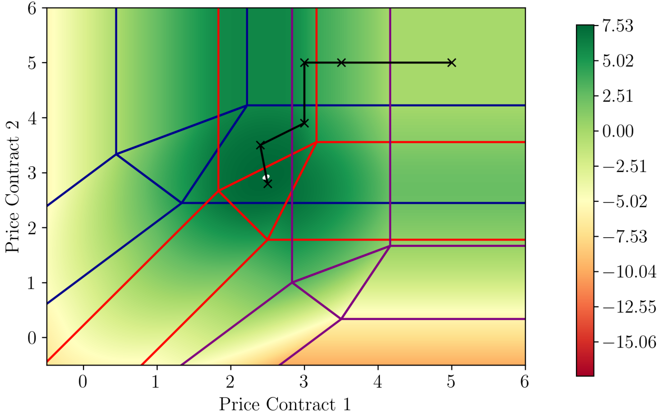


Figure: Example with $K = 3$ segments and $N = 3$ contracts

QPCC reformulation

The follower problem is convex, and can be replaced by KKT conditions:

$$\max_{x \in \mathcal{X}, \mu, \eta} \quad \sum_{k \in [K]} \rho_k \eta_k + \rho_k \langle R_k - C_k, \mu_k \rangle_N - 2\beta^{-1} \rho_k \|\mu_k\|_N^2$$

$$\text{s. t.} \quad \boxed{\begin{array}{l} 0 \leq \mu_{kn} \perp 2\beta^{-1}\mu_{kn} - U_{kn}(x) - \eta_k \geq 0, \ \forall k, n \\ 0 \leq \mu_{kN} \perp 2\beta^{-1}\mu_k - \eta_k \geq 0, \ \forall k \\ \mu_k \in \Delta_N, \ \forall k \end{array}}$$

This leads to a convex *Quadratic Program under Complementarity Constraints* (QPCC)[1][2]

Replace the complementarity constraints by Big-M constraints
$\rightsquigarrow$ *MIQP formulation* (that can be directly solved by CPLEX for example).

[1] L. Bai, J. Mitchell, and J.-S. Pang. "On convex quadratic programs with linear complementarity constraints". In: *Computational Optimization and Applications* 54 (Apr. 2013)

[2] F. Jara-Moroni, J. Mitchell, J.-S. Pang, and A. Wächter. "An enhanced logical benders approach for linear programs with complementarity constraints". In: *Journal of Global Optimization* 77 (May 2020)

Numerical Results

- Up to 50 segments
- Up to 10 contracts

Resolution with several methods

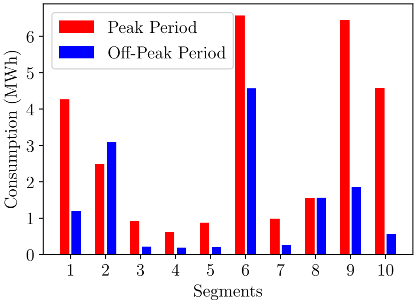
	Det.	MIQP (CPLEX)	Black-box (CMA-ES[1])	NLP (FilterMPEC[2])	*Our approach*
Time	$< 10s$	$> 1h$	$\sim 230s$	$\sim 15s$	$\sim 100s$
Variance	-	-	up to 8%	-	$< 1\%$
Optimum	Gap : 1%	Gap : 3%	up to 1% of best	up to 5% of best	best known

[1] N. Hansen. "The CMA evolution strategy: a comparing review". In: *Towards a new evolutionary computation. Advances on estimation of distribution algorithms*. New York: Springer, 2006, pp. 75–102

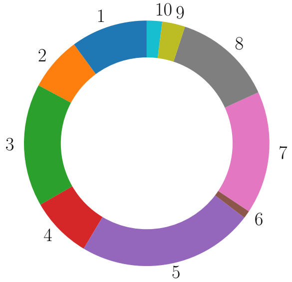
[2] R. Fletcher and S. Leyffer. FilterMPEC. Available at https://neos-server.org/neos/solvers/cp:filterMPEC/AMPL.html

Test case (1)

1 2	Base Peak/Off peak	Standard	Low cost offers (digital-only customer services)
3 4	Base Peak/Off peak	Green	Higher costs, but preferred by some segments (higher reservation bill)

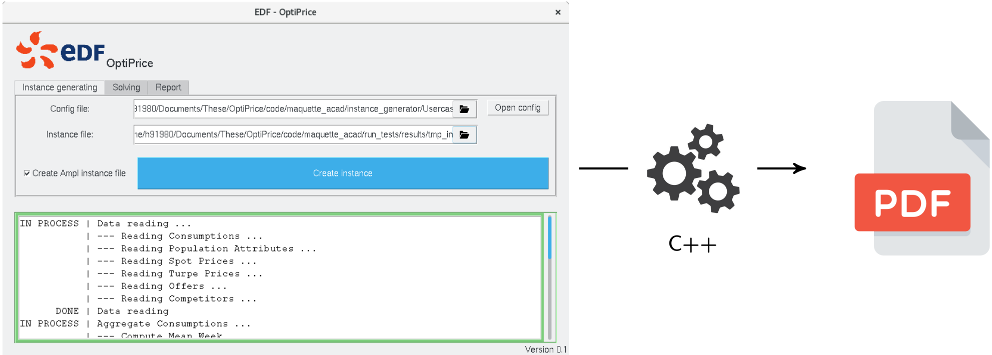


(a) Nominal consumption of segments, over one year. For each segment, the consumption is separated into the Peak period and the Off-peak period.



(b) Weights of segments. For each segment, the size of the section corresponds to the proportion of users in this segment.

Test case (2)


EDF - OptiPrice
eDF OptiPrice
Instance generating
Solving
Report
Config file:
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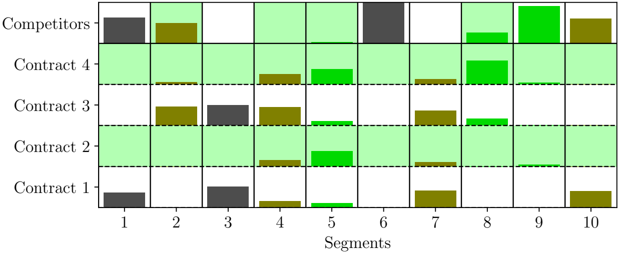
Open config
Instance file:
ne/h91980/Documents/These/OptiPrice/code/maquette_acad/run_tests/results/tmp_in
Create Ampl instance file
Create instance
IN PROCESS | Data reading ...
| --- Reading Consumptions ...
| --- Reading Population Attributes ...
| --- Reading Spot Prices ...
| --- Reading Turpe Prices ...
| --- Reading Offers ...
| --- Reading Competitors ...
DONE | Data reading
IN PROCESS | Aggregate Consumptions ...
Version 0.1
C++
PDF

Test case (2)

Optimal prices (Upper decision)

Contract	1	2	3	4
Peak (€/kWh)	0.1693	0.1834	0.1863	0.1895
Off peak (€/kWh)			0.1491	0.1626
Fixed portion (€)	133.7	129.29	122.95	128.19

Customers distribution[1] (Lower decision)



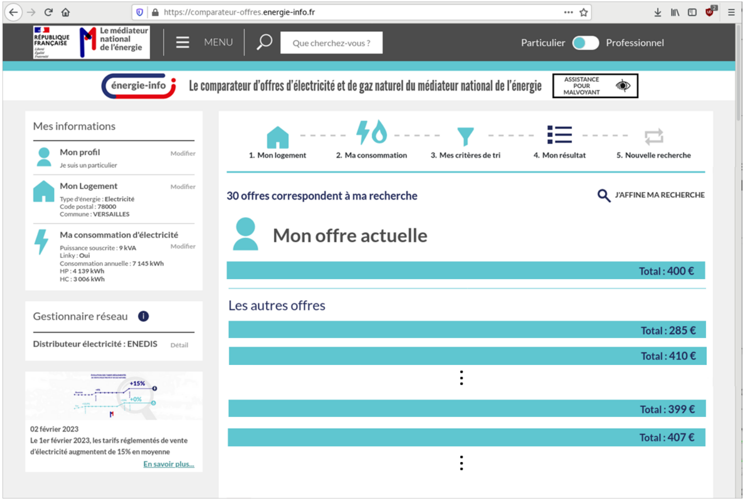
[1] Optimal customers' distribution with quadratic regularization of intensity $\beta = 0.2$. The size of the bar defines the probability of choices, i.e., a bar taking a fourth of the rectangle height represents a choice probability of 25%.

Optimal control

IMPACT OF SWITCHING COSTS

Q. J., W. van Ackooij, C. Alasseur, and S. Gaubert. "Ergodic control of a heterogeneous population and application to electricity pricing". In: *2022 IEEE 61st Conference on Decision and Control (CDC)*. 2022

The consumer' decision *at time t*



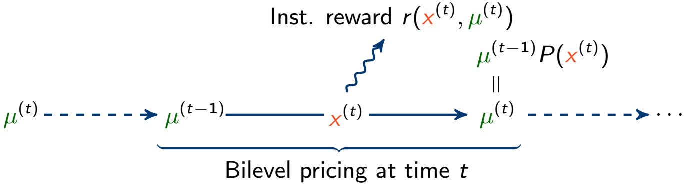
Offers of my current provider

Offers of other providers

Figure: Example of price comparison engine (French electricity market)

Switching costs

High-level description as lifted MDP[2]



1. *Distribution*: $\mu_k^{(t)} \in \Delta_N$ the distribution of the population of cluster k over $[N]$.
2. *Instantaneous reward*: $r : (x^{(t)}, \mu^{(t)}) \mapsto \sum_{k \in [K]} \rho_k \left\langle \theta_k(x^{(t)}), \mu_k^{(t)} \right\rangle_N$ ← upper objective at time t
3. *(Linear) Transition*: $\mu_k^{(t)} = \mu_k^{(t-1)} P_k(x^{(t)})$ ← lower decision at time t
4. *Leader's (global) objective* (average long-term reward):

$$g^*(\mu^{(0)}) = \sup_{\pi \in \Pi} \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r(\pi_t(\mu^{(t)}), \mu^{(t)}) \ . \tag{AvR}$$

[2] M. Motte and H. Pham. "Mean-field Markov decision processes with common noise and open-loop controls". In: *The Annals of Applied Probability* 32.2 (Apr. 2022)

Specification to the Electricity Market context

Main example: The transition probability follows a *logit response*[1]:

$$[P_k(x)]_{n,m} = \frac{e^{\beta[U_{km}(x)+\gamma_{kn}\,\mathbb{1}_{m=n}]}}{\sum_{l\in[N]} e^{\beta[U_{kl}(x)+\gamma_{kn}\,\mathbb{1}_{l=n}]}} > 0 \ ,$$

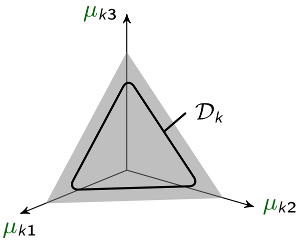
- γ_{kn} is the cost for segment k to *switch* from contract n to another one,
- β is the intensity of the choice (it can represent a "*rationality* parameter").

Link with static model: if a representative agent chooses the contract n at time $t-1$, then

$$\mu_k^{(t)} \in \arg\max_{\mu\in\Delta_N} \left\{ \left\langle U_k(x^{(t)}) + \gamma_{kn}\,\mathbb{1}_{\cdot=n}, \mu_k^{(t)} \right\rangle_N - \frac{1}{\beta}\langle \log(\mu_k), \mu_k\rangle_N \right\}$$

[1] P. Pavlidis and P. B. Ellickson. "Implications of parent brand inertia for multiproduct pricing". In: *Quantitative Marketing and Economics* 15.4 (July 2017), pp. 369–407

Ergodic control



Let $\mathcal{D}_k := \operatorname{vex}(\{\mu_k P_k(x) \mid x \in \mathcal{X}, \mu_k \in \Delta_N\})$,
and $\mathcal{D} = \bigtimes_{k \in [K]} \mathcal{D}_k$.

Lemma

$\mathcal{D}_k \subseteq \operatorname{relint} \Delta_N^K$.
Moreover, for $t \geq 1$, $\mu^{(t)} \in \mathcal{D}$ for any policy $\pi \in \Pi$.

For $v : \Delta_N^K \to \mathbb{R}$, the *Bellman operator* $\mathcal{B}$ is

$$\mathcal{B}\, v\,(\mu) = \max_{x \in \mathcal{X}} \{r(x, \mu) + v(\mu P(x))\} \ .$$

Theorem

The *ergodic eigenproblem*

$$g\, \mathbb{1}_{\mathcal{D}} + h = \mathcal{B}\, h$$

admits a solution $g^* \in \mathbb{R}$ and h^* Lipschitz and convex on $\mathcal{D}$.
Moreover, g^* satisfies (AvR), and $x^*(\cdot) \in \arg\max \mathcal{B}\, h^*$ defines an *optimal policy*.

Deterministic MDP without controllability – the most degenerate case

	Time	Transitions	Assumption
[Sch85]	discrete	stochastic	unichain[3]
[Bis15]	discrete	stochastic	Doeblin / minorization[4]
[MN02]	discrete	deterministic	quasi-compactness
[Fat08]	continuous	deterministic	controlability[5]
[Zav12]	discrete	deterministic	controlability
[CGG14]	continuous	deterministic	contraction of the dynamics (*A*2)
This work	discrete	deterministic	contraction of the dynamics (*A*2)

} weak-KAM

Standard unichain/Doeblin type conditions entail that the eigenvector is *unique*, up to an additive constant, this is *no longer true* in our case.

[3] the Markov Chain induced by any deterministic stationary policy consists of a single recurrent class plus a –possibly empty– set of transient states (i.e., there exists a subset of states that are visited infinitely often with probability 1 independently of the starting state)

[4] for all state s, action a and measurable subset B of the state space, $P(B|x,a) \geq \epsilon\mu(B)$

[5] for every pair of states (s, s'), there exists an action a making s' accessible from s

Ergodic control – Sketch of the proof (existence)

We use a contraction argument directly on the dynamics (*not on* the Bellman Operator):
Let d_H be the Hilbert's projective metric defined as

$$d_H(u,v) = \max_{1\leq i,j\leq n} \log\left(\frac{u_i}{v_i}\frac{v_j}{u_j}\right) \ .$$

$(\mathcal{D}, d_H)$ is a complete metric space.

Birkhoff theorem

Every matrix $Q \gg 0$ is a contraction in Hilbert's projective metric, i.e.,

$$\forall \mu,\nu \in (\mathbb{R}^N_{>0}),\ d_H(\mu Q, \nu Q) \leq \kappa_Q d_H(\mu,\nu) \ ,$$

where $\kappa_Q := \tanh\left(\mathrm{Diam}_H(Q)\,/\,4\right) < 1$.

We then use the method of *vanishing discount approach*[1]:

→ the family of α-discounted objective function $(V_\alpha(\cdot))_\alpha$ is *equi-Lipschitz*, which entails the existence of the eigenvector by a *compactness* argument.

[1] P.-L. Lions, G. Papanicolaou, and S. Varadhan. "Homogenization of Hamilton-Jacobi equation". Jan. 1987

Policy Iteration

- Regular grid $\Sigma = (\hat{\mu}_{\vec{i}})_{\vec{i} \in [M]^K}$ of the simplex Δ_N^K,
- Bellman Operator $\mathcal{B}^{\Sigma}$ using semi-lagrangian discretization (closest neighbor).

Algorithm Policy Iteration with on-the-fly transition generation

Require: Local grid Λ, local transitions $(T^{\Lambda,k})_{k \in [K]}$, initial decision vector $\hat{d}'$

do

$\hat{d} \leftarrow \hat{d}'$

$\hat{g}, \hat{h}$ solution of $\begin{cases} \hat{g} + \hat{h}_{\vec{i}} = r(\hat{d}_{\vec{i}}, \hat{\mu}_{\vec{i}}) + \hat{h}_{\vec{j}}, \ \vec{i} \in \Sigma \\ \vec{j} = T^{\Sigma}(\vec{i}, \hat{d}_{\vec{i}}) \end{cases}$ ▷ Policy Evaluation

for $\vec{i} \in \Sigma$ **do**

$\hat{d}'_{\vec{i}} \leftarrow \arg\min_{x \in \mathcal{X}} \left\{ r(x, \hat{\mu}_{\vec{i}}) + \hat{h}_{\vec{j}} \text{ s.t. } \vec{j} = T^{\Sigma}(\vec{i}, x) \right\}$ ▷ Policy Improvement

end for

while $\hat{d}' \neq \hat{d}$

return $\hat{g}, \hat{d}$

[1] J. Cochet-Terrasson, G. Cohen, S. Gaubert, M. McGettrick, and J.-P. Quadrat. "Numerical Computation of Spectral Elements in Max-Plus Algebra". In: *IFAC Proceedings Volumes* 31.18 (July 1998), pp. 667–674

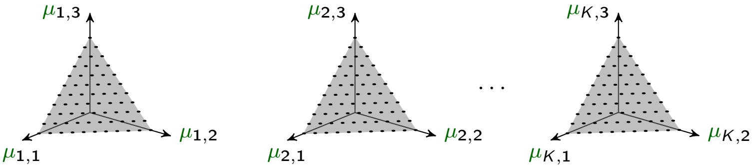
Policy Iteration

- Regular grid $\Sigma = (\hat{\mu}_{\vec{i}})_{\vec{i} \in [M]^K}$ of the simplex Δ_N^K,
- Bellman Operator $\mathcal{B}^{\Sigma}$ using semi-lagrangian discretization (closest neighbor).
- *On-the-fly generation* of transitions, refining the combinatorial version of Howard's scheme[1].

$$\vec{j} = T^{\Sigma}(\vec{i}, x)$$

Decomposition (ind. populations)

$$\vec{j}_1 = T^{\Lambda,1}(\vec{i}_1, x) \qquad \vec{j}_2 = T^{\Lambda,2}(\vec{i}_2, x) \qquad \vec{j}_K = T^{\Lambda,K}(\vec{i}_K, x)$$



[1] J. Cochet-Terrasson, G. Cohen, S. Gaubert, M. McGettrick, and J.-P. Quadrat. "Numerical Computation of Spectral Elements in Max-Plus Algebra". In: *IFAC Proceedings Volumes* 31.18 (July 1998), pp. 667–674

Numerical results

Instance	(node, arcs)	RVI (with K.-M. damping)	PI (combinatorial)	*This work*
$K = 1, N = 1$ $\delta_\mu = 1/2000$	(2e3, 2.5e6)	70s 0.8Mo	1s 30Mo	0.2s 9Mo
$K = 2, N = 2$ $\delta_\mu = 1/50$	(7.4e5, 6.9e8)	7h 15Mo	390s 13Go	70s 103Mo

Table: Comparison with combitorial Howard algorithm[1] and RVI with Krasnoselskii-Mann damping[2,3].

[1] J. Cochet-Terrasson, G. Cohen, S. Gaubert, M. McGettrick, and J.-P. Quadrat. "Numerical Computation of Spectral Elements in Max-Plus Algebra". In: *IFAC Proceedings Volumes* 31.18 (July 1998), pp. 667–674

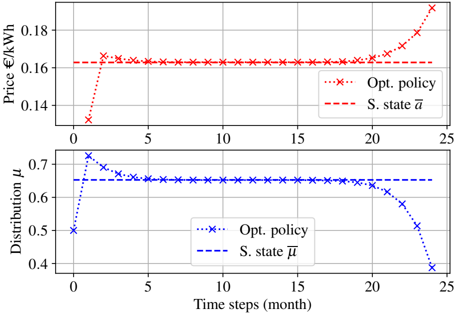
[2] A. Federgruen, P. Schweitzer, and H. Tijms. "Contraction mappings underlying undiscounted Markov decision problems". In: *Journal of Mathematical Analysis and Applications* 65.3 (Oct. 1978), pp. 711–730

[3] M. Akian, S. Gaubert, U. Naepels, and B. Terver. *Solving irreducible stochastic mean-payoff games and entropy games by relative Krasnoselskii-Mann iteration*. 2023

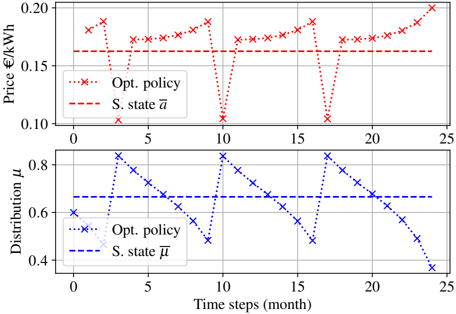
Impact of switching costs γ on toy model

"Turnpike" like strategy:
Attraction to a steady-state

Cyclic strategy:
A promotion is periodically applied



(a) Optimal finite horizon trajectory (provider action and customer distribution) for *low* switching cost.



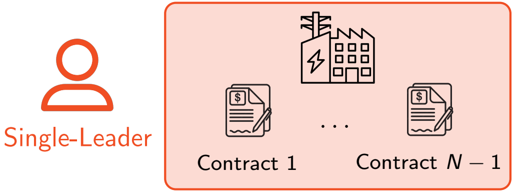
(b) Optimal finite horizon trajectory (provider action and customer distribution) for *high* switching cost.

$\hookrightarrow$ Confirms *optimality of periodic promotions*, already observed in Economics

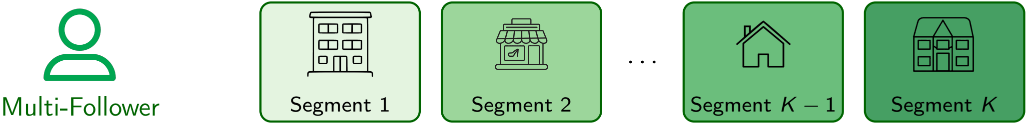
IMPACT OF THE SIZE OF THE MENU

Q. J., W. van Ackooij, C. Alasseur, and S. Gaubert. “A Quantization Procedure for Nonlinear Pricing with an Application to Electricity Markets”. To appear in: *2023 IEEE 62nd Conference on Decision and Control (CDC)*

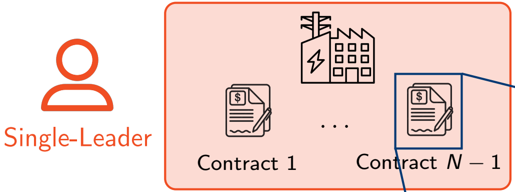
Evolutions in the model



Inmpact of the size of the menu (N) ?



Evolutions in the model



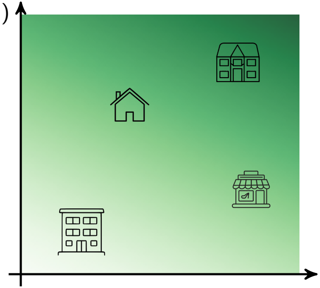
Contract structure:

$$x = \big(\overbrace{p}^{\text{Fixed portion (€)}}, \underbrace{q_1, q_2, \ldots, q_D}_{\text{Variable portions (€/kWh)}}\big)$$

Evolutions in the model

Continuum of Followers

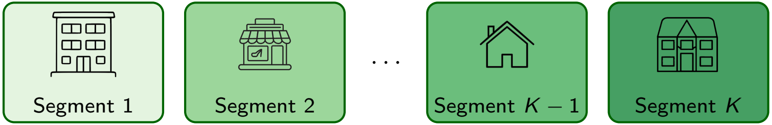
$$\int_{\mathcal{E}} \rho(e)\mathrm{d}e = 1$$



Each agent is defined by a vector of *characteristics* $e \in \mathcal{E} \subseteq \mathbb{R}^D_{\geq 0}$.

Multi-Follower

$$\sum_{k \in [K]} \rho_k = 1$$



The Monopolist problem[1]

Assumption: (Continuum of offers).
The leader constructs a *continuum* of offers, where each offer is *especially designed* for a type $e \in \mathcal{E}$:

$$(p_i, q_i)_{1 \leq i < N} \rightsquigarrow (p(e), q(e))_{e \in \mathcal{E}} \quad .$$

Optimality at the lower level:
The leader ensures that $(p(e), q(e))$ is selected by e by an *Incentive-compatibility condition* :

$$u(e_2) - u(e_1) \geq \langle e_1 - e_2, q(e_1) \rangle \, , \; \forall e_1, e_2 \in \mathcal{E} \quad , \qquad (IC)$$

with $u(e) = -p - \langle q(e), e \rangle$.

Exemple with "Tarif Bleu" ($D = 2$)

(IC) condition $\iff$ for a consumption e_2, $\underbrace{p(e_2) + \langle e_2, q(e_2) \rangle}_{\text{Invoice with contract } e_2} \leq \underbrace{p(e_1) + \langle e_2, q(e_1) \rangle}_{\text{Invoice with contract } e_1}$

(contract e_2 *really preferred* by agent e_2 compared to any other contract e_1).

---

[1] J.-C. Rochet and P. Choné. "Ironing, sweeping, and multidimensional screening". In: *Econometrica* (1998), pp. 783–826

A Convex Pricing Problem

The aim of the *monopolist* is then to maximize a revenue function, defined as

$$J(u, q) := \int_{\mathcal{E}} L(e, u(e), q(e))\mathrm{d}e - C\left(\int_{\mathcal{E}} M(e, q(e))\mathrm{d}e\right) , \tag{1}$$

In addition to (IC), $u(e)$ must be greater than a reservation utility:

$$u(e) \geq R(e) \ . \tag{IR}$$

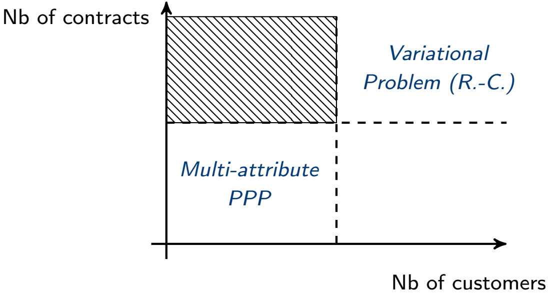
The problem solved by the monopolist is then

$$\max_{u,q} \left\{ J(u, q) \ \middle| \begin{array}{l} u, q \text{ satisfy } (IC), (IR) \\ (u(e), q(e)) \in U_e \times Q \text{ for } e \in \mathcal{E} \end{array} \right\} \tag{R.-C.}$$

Theorem

If L is *linear*, M is *strictly convex* in q, and C is *increasing* and *strictly convex*, then Problem ($R.$-$C.$) has a unique optimal solution.

Objective: Quantization of the menu of contracts



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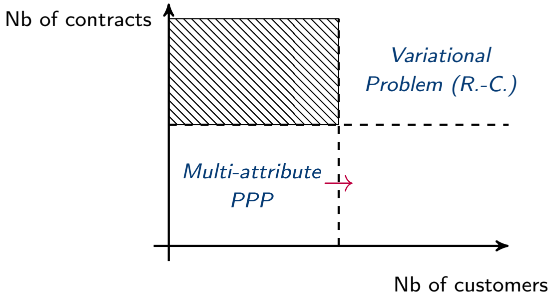
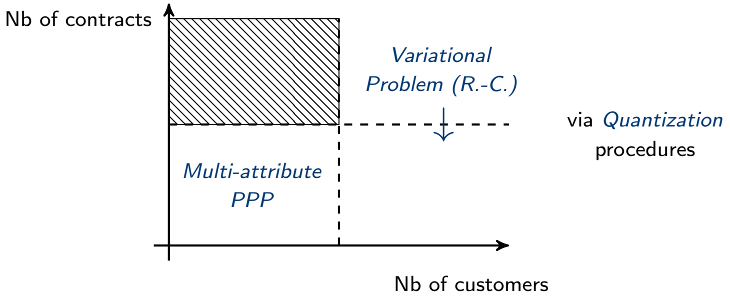


Difficulty:
The multi-attribute PPP problem with elasticity (big-M formulation) is already challenging for more than 10 customers.

Objective: Quantization of the menu of contracts



Alternative approach[a]:
Find the "best" approximation of the infinite-size menu of offers by a (small) prescribed number of contracts, i.e.,

$$\text{Approximate} \quad (p(e), q(e))_{e\in\mathcal{E}} \quad \text{by } N \text{ contracts} \quad (\hat{p}_i, \hat{q}_i)_{1\le i\le N} \ .$$

[a] D. Bergemann, E. Yeh, and J. Zhang. "Nonlinear pricing with finite information". In: *Games and Economic Behavior* 130 (Nov. 2021), pp. 62–84

“Quantization" of the utility function

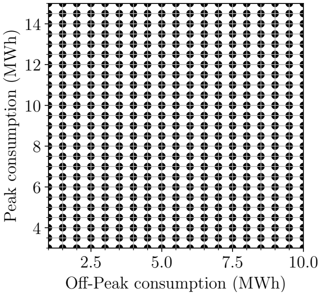
Step 1: Solve Problem (R.-C.)

- Solve the problem on a discretization grid Σ of $\mathcal{E}$[1].
- We obtain a *discretized infinite-size menu* $(\hat{p}_i, \hat{q}_i)_{i\in\Sigma}$.

  The utility $\hat{u}_\Sigma$ is then defined as

$$\hat{u}_S(e) = \bigvee_{i\in S} \hat{u}_i(e)\ , \quad S \subseteq \Sigma\ ,$$

where $\hat{u}_i : e \in \mathcal{E} \mapsto -\langle \hat{q}_i, e\rangle_D - \hat{p}_i$ (“*basis function*")



[1] e.g., G. Carlier and X. Dupuis. “An iterated projection approach to variational problems under generalized convexity constraints”. In: *Applied Mathematics and Optimization* 76.3 (2017), pp. 565–592

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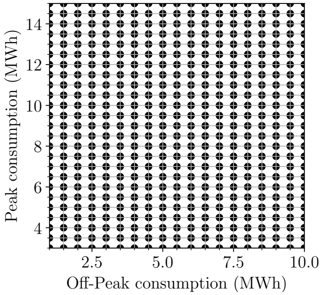


Step 2: Select from the $|\Sigma|$ contracts the N "best" contracts

$$\min_{S\subseteq\Sigma} \{\text{"Distance"}(\hat{u}_S, \hat{u}_\Sigma) \text{ s.t. } |S| \le N\} \;. \tag{2}$$

[1] e.g., G. Carlier and X. Dupuis. "An iterated projection approach to variational problems under generalized convexity constraints". In: *Applied Mathematics and Optimization* 76.3 (2017), pp. 565–592

Importance metric

$$\min_{S \subseteq \Sigma} \{ d(\hat{u}_S, \hat{u}_\Sigma) \text{ s.t. } |S| \leq N \} \quad , \tag{3}$$

1. *L_∞ (resp. L_1) norm*: $d_\infty(u, v) = \|u - v\|_{L_\infty(X)}$ (resp. $d_1(u, v) = \|u - v\|_{L_1(X)}$),
2. *J-based criterion*: $d_J(u, v) = J(v, q_v) - J(u, q_u)$. ($\leftrightarrow$ maximization of revenue)[6].

Definition (Importance metric)[7]

$$\nu(S, i) = d(\hat{u}_{S \setminus \{i\}}, \hat{u}_S) \quad . \tag{4}$$

This corresponds to an *incremental version* of the criteria (3).

$\rightarrow$ (L_∞ / L_1): it expresses the *difference between the "shape"* of $\hat{u}_S$ with and without $\hat{u}_i$

$\rightarrow$ (J-based): it expresses the *loss of revenue* when contract i is removed.

---

[6] $q_u := -\nabla u$, see J.-C. Rochet and P. Choné. "Ironing, sweeping, and multidimensional screening". In: *Econometrica* (1998), pp. 783–826

[7] W. M. McEneaney, A. Deshpande, and S. Gaubert. "Curse-of-complexity attenuation in the curse-of-dimensionality-free method for HJB PDEs". In: *2008 American Control Conference*. IEEE, June 2008

Greedy descent approach

"One-shot procedure"	[MDG08]	Sort the importance metric and *keep the n "most important"* basis functions.
"Greedy ascent approach"	[GMQ11]	Iteratively *add the "most important"* basis function to S.
"Bundle-based pruning"	[GQS14]	Introduction of bundle methods for time reduction.

Here, *Greedy descent approach*:

(i) $S \leftarrow \Sigma$

(ii) While $|S| > n$,

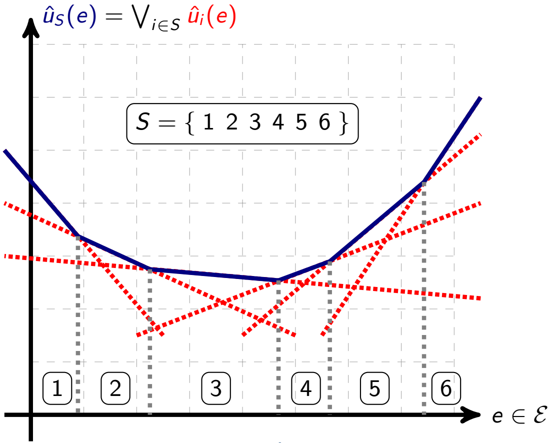
1. For each $i \in S$, compute $\nu(S, i)$.
2. Sort the importance metric and *remove the "least important"* basis function.

This pruning problem is a continuous version of the facility location problem[1] (NP-Hard).

Pros: More accurate pruning (reduction of the approximation error)
Cons: More time consuming (recomputation of the importance metric at each step)

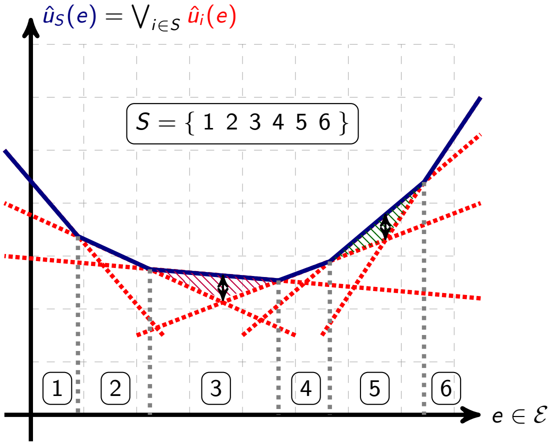
[1] S. Gaubert, W. McEneaney, and Z. Qu. "Curse of dimensionality reduction in max-plus based approximation methods: Theoretical estimates and improved pruning algorithms". In: *IEEE Conference on Decision and Control and European Control Conference*. IEEE, Dec. 2011

$1D$ Example



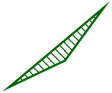
Maximization diagram :
Subdivision of $\mathcal{E}$ in cells
$V_i = \{e \in \mathcal{E} \mid \hat{u}_i(e) \geq \hat{u}_j(e),\ \forall j \in S\}$

$1D$ Example



L_1 *importance metric* :

$\nu(S,3) = \mathcal{A}\Big($  $\Big)$

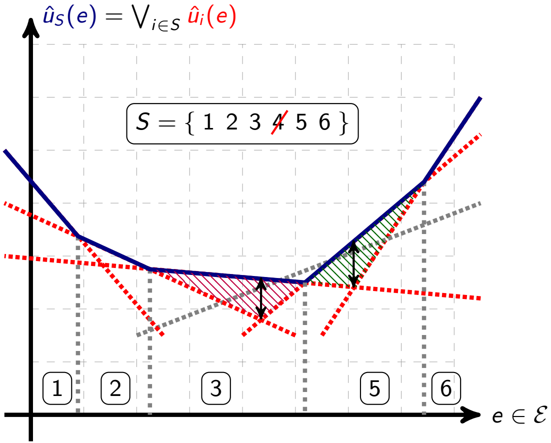
$\nu(S,5) = \mathcal{A}\Big($  $\Big)$

L_∞ *importance metric* :

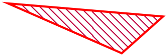
$\nu(S,3) =$ 

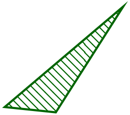
$\nu(S,5) =$ 

$1D$ Example



L_1 importance metric :

$\nu(S,3) = \mathcal{A}\Big($  $\Big)$

$\nu(S,5) = \mathcal{A}\Big($  $\Big)$

L_∞ importance metric :

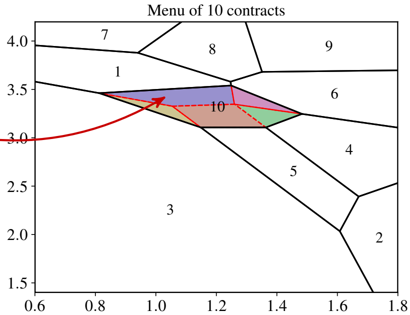
$\nu(S,3) =$ 

$\nu(S,5) =$ 

Key point : When $\hat{u}_4$ is removed, *only* $\nu(S,3)$ and $\nu(S,5)$ *change* (neighboring cells).

L_1 and J-based case

The blue polyhedron corresponds to $F_{1,-10} \cap V_{10}$



Customers decision as a Maximization diagram (polyhedral complex):

For a set S of contracts,

- $V_i = \{e \in \mathcal{E} \mid \hat{u}_i(e) \geq \hat{u}_j(e),\ \forall j \in S\}$ (= customers who *choose contract* i),
- $F_{j,-i}$ is the *future* cell of j if i is removed, i.e., $F_{j,-i} = \{e \in \mathcal{E} \mid \hat{u}_j(e) \geq \hat{u}_k(e),\ \forall k \neq i \in S\}$

Three routines are used:

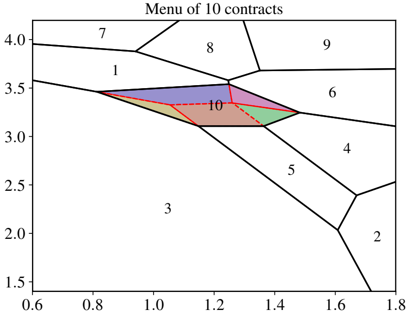
- $\textsc{Vrep}(S, i)$ returns the representation by vertices of V_i (reverse search algorithm `lrs`),
- $\textsc{updateNeighbors}$ updates the neighbors of each cell knowing the vertex representation,
- $\textsc{updateImpMetric}$ updates $\nu(S, i)$ for all $i \in I$.

L_1 and J-based case

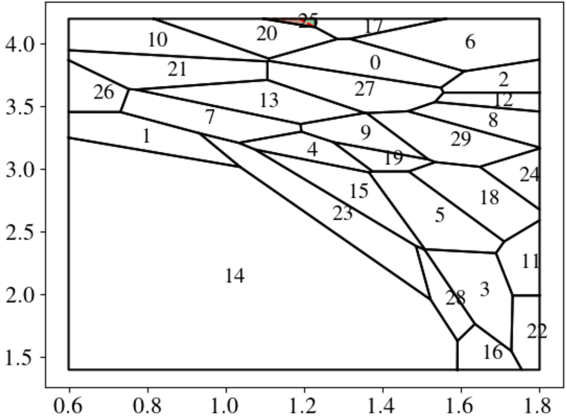
Algorithm 2: Pruning with *local update*

Require: N

for $i \in \Sigma$ **do**
  $V_i \leftarrow \text{VREP}(\Sigma, i)$ ▷ Initial Vertex representation
end for
$S \leftarrow \Sigma$
$I \leftarrow \Sigma$ ▷ Index of problems to recompute
for $t = 1 : |\Sigma| - N$ **do**
  $(J_i)_{i \in I} \leftarrow \text{UPDATENEIGHBORS}((V_i)_{i \in I})$
  for $i \in I$, $j \in J_i$ **do**
    $F_{j,-i} \leftarrow \text{VREP}(S \backslash \{i\}, j)$ ▷ Future cells
  end for
  $\nu \leftarrow \text{UPDATEIMPMETRIC}(I, (V_i)_{i \in S}, (F_{j,-i})_{j \in J_i, i \in S})$
  $r \leftarrow \arg\min_{i \in S} \nu_i$ ▷ Contract to remove ("least important" one)
  $S \leftarrow S \backslash \{r\}$
  for $j \in J_r$ **do**
    $V_j \leftarrow F_{j,-r}$ ▷ Update Vertex representation
  end for
  $I \leftarrow J_r$
end for
return S



Algorithm example



Complexity results

Proposition

The importance metric of a contract $i \in S$ stays *unchanged* when we remove a contract j which is not in the neighborhood of i, i.e., $\nu(S \backslash \{j\}, i) = \nu(S, i)$ for $j \in S \backslash J_i$.

Proposition (Critical steps)

Suppose that $|J_i| \leq m$ (*maximum number of neighbors* of a cell during the execution).

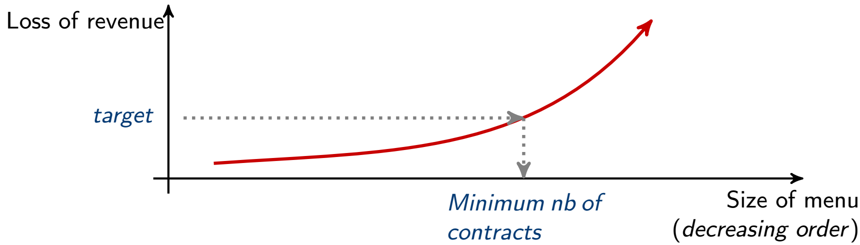
$$\# \text{ calls to } \textsc{VRep}(S, i)$$

$$O(m|\Sigma|^2) \quad \rightsquigarrow \quad O(m^2|\Sigma|)$$

Remark: reverse search has an incremental running time of $O(|\Sigma| d)$ per vertex if the input is nondegenerate[1].

[1] D. Avis. "A Revised Implementation of the Reverse Search Vertex Enumeration Algorithm". In: *Polytopes — Combinatorics and Computation*. Ed. by G. Kalai and G. M. Ziegler. Basel: Birkhäuser Basel, 2000, pp. 177–198

Numerical results



Objective of the retailer:

Finding the *minimum number of contracts* needed to obtain a loss of revenue *lower than a target*.

Numerical results

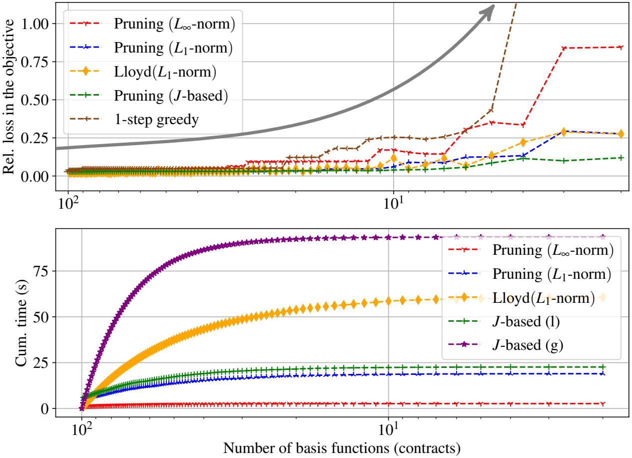


Figure: Comparison of error bounds.
(g) stands for global update while (l) stands for local update.

Other contributions

- ◇ **Chapter 7: Principal-Multi-Agent model**[1]
  Design of a rank-based reward for energy savings purposes.
- ◇ **Chapter 8: Chance-Constrained Programming**[2]
  Study of distributionally robust models using Bennett-type concentration inequalities.
- ◇ **Chapter 9: Sparse optimization**[3]
  Study of entropic lower bounds for sparse optimization using Schur convexity.

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[1] C. Alasseur, E. Bayraktar, R. Dumitrescu, and Q. J. *A Rank-Based Reward between a Principal and a Field of Agents: Application to Energy Savings*. preprint. 2022

[2] Q. J. and R. Zorgati. *Tight Bound for Sum of Heterogeneous Random Variables: Application to Chance Constrained Programming*. 2022

[2] Q. J., A. Bialecki, L. E. Ghaoui, S. Gaubert, and R. Zorgati. "Entropic Lower Bound of Cardinality for Sparse Optimization". Nov. 2022

Perspectives

- ◇ **Elasticity of the demand**:
  $\rightarrow$ Extend to more general cases than iso-elasticity.

- ◇ **Link between turnpike properties and weak-KAM theory**:
  $\rightarrow$ Extend the results of convergence to Aubry set (using strict-dissipativity) to non-controllable cases.

- ◇ **Partial participation**:
  $\rightarrow$ Extend the quantization methods to partial participation of the consumers.

- ◇ **Bounds for the approximation error made with the quantization approach**:
  $\rightarrow$ Classical approximation results do not apply in our context.

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Thank you for your attention

Questions ?

KKT transformation

The follower problem is linear, and can be replaced by KKT conditions:

$$\max_{x \in \mathcal{X}, \mu, \eta} \quad \sum_{k \in [K]} \rho_k \eta_k + \rho_k \langle R_k - C_k, \mu_k \rangle_N$$

$$\text{s.t.} \quad \begin{aligned} & 0 \leq \mu_{kn} \perp U_{kn}(x) + \eta_k \leq 0, \ \forall k, n \\ & 0 \leq \mu_{kN} \perp \eta_k \leq 0, \ \forall k \\ & \mu_k \in \Delta_N, \ \forall k \end{aligned}$$

This leads to a *Linear Program under Complementarity Constraints* (LPCC).

Usually, compl. constraints replaced by Big-M constraints $\rightsquigarrow$ *MILP formulations*[1][2]

[1] R. Shioda, L. Tunçel, and T. Myklebust. "Maximum utility product pricing models and algorithms based on reservation price". In: *Computational Optimization and Applications* 48 (Mar. 2011), pp. 157–198

[2] C. G. Fernandes, C. E. Ferreira, A. J. Franco, and R. C. Schouery. "The envy-free pricing problem, unit-demand markets and connections with the network pricing problem". In: *Discrete Optimization* 22 (2016), pp. 141–161

Impact of the regularization intensity

$\beta = 0$ — *Random* choice

$\beta = \infty$ — Deterministic case

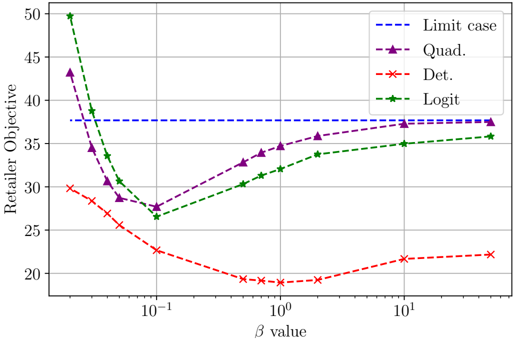
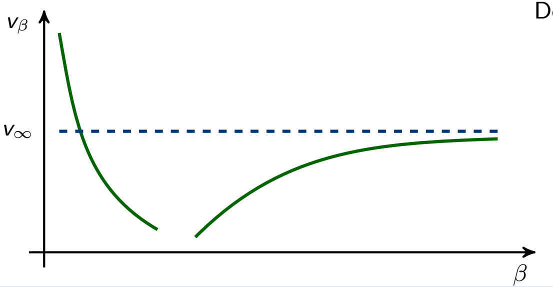


Figure: Optimal value as a function of the rationality parameter β.
'Logit': model under logit response, 'Quad.': model under quadratic response
'Det': objective value obtained with the optimal deterministic prices but under quadratic response.

Impact of the regularization intensity

Random choice

Deterministic case



$v_\infty = \max_n (R_n - C_n)$

$\# v_\infty = |\arg\max_n (R_n - C_n)|$

Theorem:

For the standard MNL model ($K = 1$),

1. $\lim_{\beta \to 0} (\beta v_\beta) = \mathcal{W}_0 \left((N\text{-}1)/e \right)$; where $\mathcal{W}_0$ denotes the Lambert function.
2. if $v_\infty > 0$ then $v_\beta \underset{\beta \to +\infty}{=} v_\infty - \frac{\ln(\beta v_\infty)}{\beta} + \frac{\ln(\# v_\infty) - 1}{\beta} + o\left(\frac{1}{\beta}\right)$.

Bilevel optimization with uncertainty[1]

Here-and-now leader		Gumbell uncertainty		Wait-and-see follower
x	$\curvearrowright$	$\widetilde{U}_{kn}(x, \varepsilon) = U_{kn}(x) + \varepsilon_{kn}$	$\curvearrowright$	$y_{kn}(x, \varepsilon) = \mathbb{1}_{(\widetilde{U}_{kn}(x,\varepsilon) > \widetilde{U}_{km}(x,\varepsilon), m \neq n)}$

Risk-neutral leader:

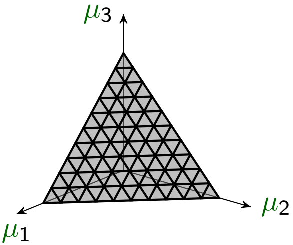
$$\max_{x \in \mathcal{X}} \mathbb{E}_\varepsilon \left[\sum_{k \in [K]} \rho_k \langle \theta_k(x), y_k^* \rangle_N \right] = \max_{x \in \mathcal{X}} \sum_{k \in [K]} \rho_k \langle \theta_k(x), \mu_k^* \rangle_N$$

where $\mu_{kn}^* = \mathbb{P}\left[\widetilde{U}_{kn}(x, \varepsilon) > \widetilde{U}_{km}(x, \varepsilon), m \neq n \right]$.

[1] Y. Beck, I. Ljubić, and M. Schmidt. "A survey on bilevel optimization under uncertainty". In: *European Journal of Operational Research* (Feb. 2023)

Relative Value Iteration with Krasnoselskii-Mann damping

- ◇ Regular grid Σ of the simplex Δ_N^K,
- ◇ Bellman Operator $\mathcal{B}^\Sigma$ using Freudenthal triangulation[1].



Algorithm RVI with Mann-type iterates

Require: Σ, $\mathcal{B}^\Sigma$, $\hat{h}_0$

Initialize $\hat{h} = \hat{h}_0$, $\hat{h}'(\mu) = \mathcal{B}^\Sigma\,\hat{h}$

while $\mathrm{Span}(\hat{h}' - \hat{h}) > \epsilon$ **do**

  $\hat{h} \leftarrow (\,\hat{h}' - \max\{\hat{h}'\}e + \hat{h}\,)/2$

  $\hat{h}'(\hat{\mu}) \leftarrow (\mathcal{B}^\Sigma\,\hat{h})(\hat{\mu})$ for all $\hat{\mu} \in \Sigma$ ▷ Update of bias

end while

$\hat{g} \leftarrow \max(\hat{h}' - \hat{h})$

return $\hat{g}, \hat{h}$

Proposition[2]

Convergence time of RVI $= O(\epsilon^{-2})$

[1] W. S. Lovejoy. "Computationally Feasible Bounds for Partially Observed Markov Decision Processes". In: *Operations Research* 39.1 (Feb. 1991), pp. 162–175

[2] S. Gaubert and N. Stott. "A convergent hierarchy of non-linear eigenproblems to compute the joint spectral radius of nonnegative matrices". In: *Mathematical Control & Related Fields* 10.3 (2020), pp. 573–590

Weak-KAM solution

Let T_c^+ be the positive Lax-Oleinick semi-group, defined as

$$T_c^+ \, h(x) := \sup_{y \in \mathcal{X}} \{h(y) - c(x,y)\} \quad . \tag{5}$$

Existence of positive weak KAM solution, case of controllable system[1]

Assume that $c(\cdot,\cdot)$ is uniformly bounded and jointly continuous. Then, the problem

$$T_c^+ h = h + g \tag{6}$$

admits a solution $h^* \in \mathrm{Vex}(\mathcal{X})$ and $g^* \in \mathbb{R}$. Moreover, any sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_{n+1} \in \arg\max T_c^+ h^*(x_n)$ for $n \in \mathbb{N}$ minimizes the average stage cost:

$$\lambda^* = \inf_{(x_n)_{n\in\mathbb{N}}} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c(x_n, x_{n+1}) \quad . \tag{7}$$

[1] M. Zavidovique. "Strict sub-solutions and Mañé potential in discrete weak KAM theory". In: *Commentarii Mathematici Helvetici* (2012), pp. 1–39

Aubry set

Aubry set

Let $h \in \mathcal{S}$ be a critical subsolution. The *Aubry set of h*, $\widetilde{\mathbb{A}}_h \in \mathcal{X}^{\mathbb{N}}$, is defined as

$$\widetilde{\mathbb{A}}_h = \left\{ (x_n)_{n\in\mathbb{N}} \mid \forall n < p, h(x_p) - h(x_n) = \sum_{k=n}^{p-1} c(x_k, x_{k+1}) + (p-n)g^* \right\} \ .$$

The Aubry set $\widetilde{\mathbb{A}}$ is then the intersection over all the critical subsolutions, i.e., $\widetilde{\mathbb{A}} = \cap_{h\in\mathcal{S}} \widetilde{\mathbb{A}}_h$. Finally, the projected Aubry set $\mathbb{A}$ refers to the projection of the Aubry set on the first component, and is given by

$$\mathbb{A} = \left\{ x_0 \mid (x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathbb{A}} \right\} \subseteq (\mathcal{X}^2)^{\mathbb{N}} \ .$$

Projected Aubry set $\leftrightarrow$ states where an optimal strategy can go through infinitely-many times.

$\rightarrow$ In particular, a τ-cycle $(x_n)_{n\in\mathbb{N}}$, where $x_{i+\tau} = x_i$ for all $i \in \mathbb{N}$, belongs to the Aubry set if $\sum_{i=1}^{\tau} c(x_k, x_{k+1}) = -\tau g^*$, i.e., it produces an optimal average long-term reward.

Therefore, Aubry sets are able to capture the "optimal support" of the dynamics.

Turnpike properties

Strict-dissipativity condition:

$$h(y) - h(x) + \alpha(\|x - x_e\|) \leq c(x, y) + g^*, \; x, y \in \mathcal{X} \tag{8}$$

Convergence to a steady-state

If (8) holds, then $\widetilde{\mathbb{A}} = \{(x_n)_{n\in\mathbb{N}}\}$ where $x_n = x_e$ for all $n \in \mathbb{N}$.

Convergence to the Aubry set

Let h^* be a positive weak KAM solution, and $x_0 \in \mathcal{X}$. We denote by $\pi^*(\cdot) \in \arg\max T_c^+ h^*$ an optimal stationary policy and $\{x_i^*\}$ the sequence of states generated by the policy π^*. Then, all the accumulation points of the sequence $\{x_i\}$ belong to the projected Aubry set $\mathbb{A}$.

Sketch of proof: exploiting the existence of a strict subsolution h_0 such that:

$$h_0(y) - h_0(x) < c(x, y) + g^* \text{ for all } (x, y) \notin \widehat{\mathbb{A}} \;\; . \tag{9}$$

L_∞ case

$$\nu(S,i) = \max_{e \in \mathcal{E}} \left\{ \max_{j \in S} \hat{u}_j(e) - \max_{j \in S \setminus \{i\}} \hat{u}_j(e) \right\} = \max_{e \in \mathcal{E}} \min_{j \in S \setminus \{i\}} \{\hat{u}_i(e) - \hat{u}_j(e)\} \quad . \tag{10}$$

Then, the importance metric can be computed by solving a *linear program* :

$$\max_{e \in \mathcal{E},\, \nu} \{\nu \quad \text{s.t} \quad \forall j \in S \setminus \{i\},\ \hat{u}_i(e) - \hat{u}_j(e) \geq \nu\} \tag{P_i^S}$$

Algorithm 1: Pruning with *local update*

Require: n
$S \leftarrow \Sigma$
$I \leftarrow \Sigma$ ▷ Problems to recompute
for $t = 1 : |\Sigma| - n$ **do**
  for $i \in I$ **do**
    $\nu_i, \lambda_i \leftarrow$ solution of (P_i^S)
    $J_i \leftarrow \{j \in S \setminus \{i\} \mid \lambda_{ij} > 0\}$
  end for
  $r \leftarrow \arg\min_{i \in S} \nu_i$
  $S \leftarrow S \setminus \{r\}$
  $I \leftarrow \{i \in S \mid r \in J_i\}$ ▷ Neighbors
end for
return S

Proposition

Let $\{\lambda_{ij}\}$ be the optimal dual variables in (P_i^S).

Then, the importance metric of i stays *unchanged* when we remove a contract j s.t. $\lambda_{ij} = 0$, or equivalently

$$\{i \mid \nu(S \setminus \{j\}, i) \neq \nu(S,i)\} \subseteq I := \{i \mid \lambda_{ij} > 0\} \quad .$$

Resolution of the discretized R.-C. problem

$$\max_{(u_i, q_i)_{i \in \Sigma}} J^{\Sigma}(u, q)$$
$$\text{s.t. } u_i \geq R_i, \ \forall i$$
$$u_i \in [u^-, u^+], \ q_i \in [q^-, q^+], \ \forall i$$
$$u_i - u_j \geq \langle e_i - e_j, q_i \rangle_2, \ \forall i, j$$

→ We look at a special case of b-convexity constraint[1].

→ The number of convexity constraint ($O(|\Sigma|^2)$) can be reduced[2] to $O(|\Sigma| \ln^2 |\Sigma|)$ in $\mathbb{R}^2$.

→ Here, we use an iterative procedure:

1. Start with $u_i - u_j \geq \langle e_i - e_j, q_i \rangle_2, \ \forall i, j$ such that $j \in \mathcal{N}(i)$.
2. Solve the discretized version with the partial set of convexity constraints.
3. If remaining convexity constraints are violated, add them to the model and return to '2.' Otherwise, return the solution.

[1] G. Carlier and X. Dupuis. "An iterated projection approach to variational problems under generalized convexity constraints". In: *Applied Mathematics and Optimization* 76.3 (2017), pp. 565–592

[2] J.-M. Mirebeau. "Adaptive, anisotropic and hierarchical cones of discrete convex functions". In: *Numerische Mathematik* 132.4 (2016), pp. 807–853

Computation of the importance metric

Exact computation of $\nu(S, i)$ in the 2D-case :

UPDATEIMPMETRIC (L_1 error)

Require: I, $(V_i)_{i\in S}$, $(F_{j,-i})_{i\in I, j\in J_i}$

for $i \in I$ **do**

  $\nu_i \leftarrow \sum_{j\in J_i} \iint_{F_{j,-i}\cap V_i} (\hat{u}_i(e) - \hat{u}_j(e))\mathrm{d}e$

end for

return ν

UPDATEIMPMETRIC (J-based error)

Require: I, $(V_i)_{i\in S}$, $(F_{j,-i})_{i\in I, j\in J_i}$

$M_0 \leftarrow \sum_{i\in S} \iint_{V_i} M(e, \hat{q}_i)\mathrm{d}x$

for $i \in S$ **do**

  $\delta_L \leftarrow \sum_{j\in J_i} \iint_{F_{j,-i}\cap V_i} L(e, \hat{u}_i(e), \hat{q}_i) - L(e, \hat{u}_j(e), \hat{q}_j)\mathrm{d}x$

  $\delta_M \leftarrow \sum_{j\in J_i} \iint_{F_{j,-i}\cap V_i} M(e, \hat{q}_j) - M(e, \hat{q}_i)\mathrm{d}x$

  $\nu_i \leftarrow \delta_L - C(M_0) + C(M_0 + \delta_M)$

end for

Green's formula

Let P a 2D-polytope describes by its vertices $(x_i, y_i) \in \mathbb{R}^2$ (counter-clockwise). Then $\forall a, b, c \in \mathbb{R}$,

$$\iint_P (ax + by + c)dxdy = \sum_{i=1}^{N} \left[\oint_{y_i}^{y_{i+1}} b(q_i + \tfrac{1}{\tau_i} y)ydy - \oint_{x_i}^{x_{i+1}} (ax + c)(p_i + \tau_i x)dx \right],$$

with $\tau_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$, $p_i := y_i - \tau_i x_i$ and $q_i := x_i - \frac{1}{\tau} y_i$.

Link with Bregman Voronoï diagrams

We define the *Bregman divergence* $D_u : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+$ with respect to a convex differentiable function u as

$$D_u(e_1, e_2) = u(e_1) - u(e_2) - \langle e_1 - e_2, \nabla u(e_2) \rangle \tag{11}$$

Definition (Bregman Voronoï diagram[1])

Let $\mathcal{S} = \{e_1, \ldots, e_n\}$ be a set of n points of $\mathcal{E}$. We call *Bregman Voronoï diagram* of $\mathcal{S}$:

$$\mathrm{vor}_u(e_i) := \{e \in \mathcal{E} \mid D_u(e, e_i) \leq D_u(e, e_j),\ \forall j \in [n]\} \ . \tag{12}$$

The point e_i, associated with the Voronoï cell $\mathcal{C}_i = \mathrm{vor}_u(e_i)$, is called a *site*.

Proposition (Interpretation as Voronoï diagram)

Let $\mathcal{S} = \{e_1, \ldots, e_n\}$ be a set of n points of $\mathcal{E}$. We define the family of function $\hat{u}_i$ as the supporting hyperplanes of u at e_i, i.e.,

$$\hat{u}_i(e) = u(e_i) + \langle e - e_i, \nabla u(e_i) \rangle \ .$$

Then, the *maximization diagram* of $\{\hat{u}_i\}_{1 \leq i \leq n}$ and the *Bregman Voronoï diagram* of $\mathcal{S}$ *coincides*.

[1] J.-D. Boissonnat, F. Nielsen, and R. Nock. "Bregman Voronoi Diagrams". In: *Discrete and Computational Geometry* 44.2 (Apr. 2010), pp. 281–307

Clustering with Bregman distance

We associate to $\mathcal{E}$ the p.d.f. ρ satisfying $\int_{\mathcal{E}} \rho(e)\mathrm{d}e$.
We denote by $L_u(\mathcal{S})$ the loss of optimality induced by a set of representatives $\mathcal{S} = \{e_1, \ldots, e_n\}$:

$$L_u(\mathcal{S}) = \sum_{i=1}^{n} \int_{\mathrm{vor}_u(e_i)} D_u(e, e_i)\rho(e)\mathrm{d}e = \int_{\mathcal{E}} (u(e) - \max_{1 \leq i \leq n} \hat{u}_i(e))\rho(e)\mathrm{d}e \tag{13}$$

If ρ is the uniform distrib., $L_u(\mathcal{S})$ is the L_1-error between $u(\cdot)$ and the upper envelope of $\{\hat{u}_i\}_{1 \leq i \leq n}$.

Algorithm 3 : Bregman Hard Clustering – Lloyd procedure ([Ban+05])

Require: number of cluster n, initial centroids $\{e_i^{(0)}\}_{1 \leq i \leq n}$

$t \leftarrow 0$
do
  $\mathcal{C}_i^{(t)} \leftarrow \{e \in \mathcal{E} \mid D_u(e, e_i^{(t)}) \leq D_u(e, e_j^{(t)}),\ \forall j \in [n]\}$ for all $i \in [n]$ ▷ Assignment step
  $e_i^{(t+1)} = \int_{\mathcal{C}_i^{(t)}} e\rho_{\mid \mathcal{C}_i^{(t)}}(e)\mathrm{d}e$ ▷ Centroid estimation step
  $t \leftarrow t + 1$
while there exist $i \in [n]$ such that $e_i^{(t)} \neq e_i^{(t-1)}$
return $\{e_i^{(t)}\}_{1 \leq i \leq n}$

Isoelasticity (1)

Details on the model :

- Each contract is defined by a *fixed price component* $p \in \mathbb{R}$ (in €), and d *variable price components* $z \in \mathbb{R}^d$ (in €/kWh) (typically $d = 2$ in France).

- The price coefficients (p, z) belong to a non-empty polytope $P \times Z \subset \mathbb{R}^{d+1}$:

$$P = [p^-, p^+], \quad Z := \left\{ z^- \leq z \leq z^+ \mid z_{i_1} \leq \kappa_{i_1,i_2} z_{i_2} \text{ for } i_1 \leq_{\mathcal{P}} i_2 \right\} \ ,$$

  where $\mathcal{P}$ is a *partially ordered set* (poset) of $\{1, \dots, d\}$, and $\leq_{\mathcal{P}}$ the ordering relation.

  → *Classically in electricity pricing* : inequalities between peak and off-peak prices.

- Each individual in the population is characterized by a *reference consumption vector* $e \in \mathbb{R}^d_{>0}$, and can deviate from it (*elasticity*).
  Here, we use *Constant Relative Risk Aversion* (CRRA,[Pin12; Ala+20]) :

$$\mathcal{U}_e : x \in \mathbb{R}^d_{\geq 0} \mapsto \frac{1}{\eta} \sum_{i=1}^{d} \beta_{ei} (x_i)^\eta, \ \eta \in (-\infty, 0) \cup (0, 1] \ , \qquad (14)$$

  where $\beta_e \in \mathbb{R}^d_{\geq 0}$ is the intensity of energy needs. The coefficient η is the *risk aversion* coefficient.

Isoelasticity (2)

Details on the model :

- For price coefficients $(p, z) \in \mathbb{R} \times \mathbb{R}^d$, a consumer e will optimize his consumption in order to maximize the *welfare function* :

$$\mathcal{U}_e^* : (p, z) \in \mathbb{R} \times \mathbb{R}^d \mapsto \max_{x \in \mathbb{R}_{\geq \mathbf{0}}{}^d} \{\mathcal{U}_e(x) - \langle x, z \rangle\} - p \ . \tag{15}$$

- If $e \in \mathbb{R}^d$ is obtained for reference prices $\check{p} \in \mathbb{R}$ and $\check{z} \in \mathbb{R}^d$, the *optimal consumption* of customer $\mathcal{E}_{ei}$ on period $i \in [d]$ is:

$$\mathcal{E}_{ei}(z) = e_i \left(z_i / \check{z}_i\right)^{\frac{-1}{1-\eta}} \geq 0 \ , \tag{16}$$

and the welfare function is given by

$$\mathcal{U}_e^*(p, z) = \left(\frac{1}{\eta} - 1\right) \sum_{i=1}^{d} e_i \check{z}_i \left(z_i / \check{z}_i\right)^{\frac{-\eta}{1-\eta}} - p \ . \tag{17}$$

Assumption : the provider is able to define *as many offers as consumers*

$$\text{(infinite-size) menu :} \qquad e \mapsto (p(e), q(e)) \in P \times Q$$

Model

Let us define the (weighted) invoice of a consumer as

$$\mathcal{L}_e : (p, z) \in \mathbb{R} \times \mathbb{R}^d \mapsto (p + \langle \mathcal{E}_e(z), z \rangle)\rho(e) \ , \tag{18}$$

where $\int \rho(e)\mathrm{d}e = 1$. The revenue maximization problem is then

$$\max_{p,z} \ \mathcal{J}^1(p, z) - \mathcal{J}^2(z) \tag{19a}$$

$$\text{s.t. } \mathcal{U}_e^*(p(e), z(e)) \geq \mathcal{U}_e^*(p(e'), z(e')), \ \forall e, e' \tag{19b}$$

$$\mathcal{U}_e^*(p(e), z(e)) \geq R(e), \ \forall e \tag{19c}$$

$$p(e) \in P, \ z(e) \in Z \tag{19d}$$

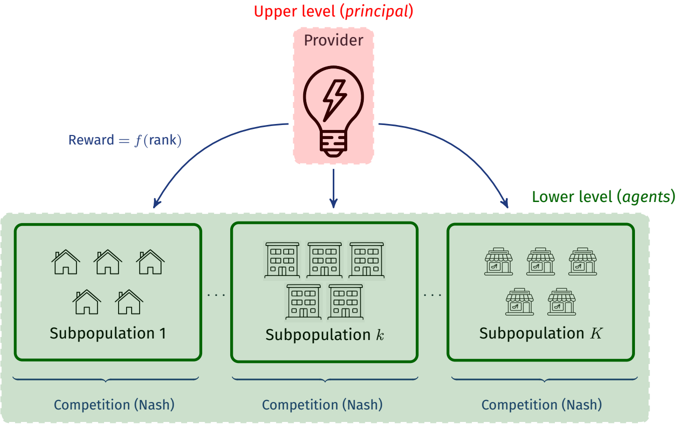
where $\mathcal{J}^1(p, z) = \int \mathcal{L}_e(p(e), z(e))\mathrm{d}e$ and $\mathcal{J}^2(z) = C\left(\int \sum_{i=1}^d \mathcal{E}_{ei}(z(e))\rho(e)\mathrm{d}e\right)$.

Recovering linear utilities : let us consider $q_i := (z_i/\check{z}_i)^{\frac{-\eta}{1-\eta}}$ .Then,

- the consumption is *convex*, expressed as $\mathfrak{E}_{ei}(q_i) = e_i[q_i]^{\frac{1}{\eta}}$,
- both the utility and the weighted invoice are linear: defining $\alpha = (\eta^{-1} - 1)\check{z}$,

$$\begin{aligned} u(e) &:= \langle e, \alpha \odot q(e) \rangle - p(e) \ , \\ L(e, u(e), q(e)) &:= \left(\frac{1}{\eta} \langle e, \check{z} \odot q(e) \rangle - u(e)\right) \rho(e) \ , \end{aligned} \tag{20}$$

Ranking game (1)

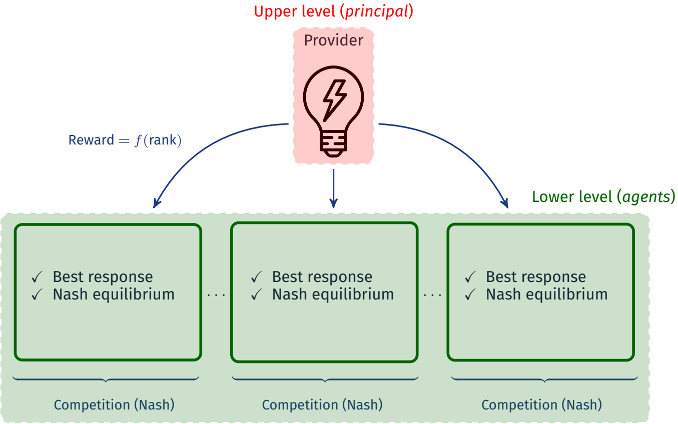


[1] R. Carmona and P. Wang. "Finite-State Contract Theory with a Principal and a Field of Agents". In: *Management Science* 67.8 (Aug. 2021), pp. 4725–4741

[2] R. Elie, T. Mastrolia, and D. Possamaï. "A Tale of a Principal and Many, Many Agents". In: *Mathematics of Operations Research* 44.2 (May 2019), pp. 440–467

[3] A. Shrivats, D. Firoozi, and S. Jaimungal. *Principal agent mean field games in REC markets*. 2021

Ranking game (2)



[1] R. Carmona and P. Wang. "Finite-State Contract Theory with a Principal and a Field of Agents". In: *Management Science* 67.8 (Aug. 2021), pp. 4725–4741

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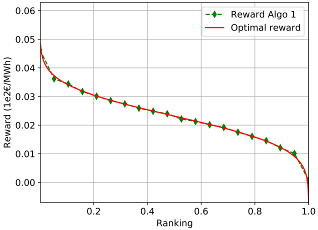
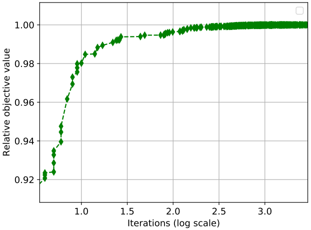
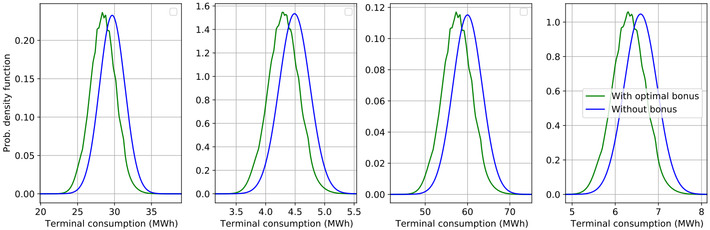
Ranking game (3)



(a) Optimal reward.



(b) Evolution of the relative objective value.



(c) Terminal consumption distribution for the four sub-populations

Benett's inequality

Refined Bennett's inequality[1]

Let $\xi_1, \ldots, \xi_N$ be N independent random variables. If there exist $b, \sigma \in \mathbb{R}^N$ such that such that

(i) $\mathbb{P}[\xi_k - \mathbb{E}[\xi_k] \leq b_k] = 1$, $k \in \{1, \ldots, N\}$,

(ii) $\mathrm{Var}(\xi_k) \leq \sigma_k^2$, $k \in \{1, \ldots, N\}$.

Then, introducing $\gamma_k := \frac{\sigma_k^2}{b_k^2}$, for all $d \geq 0$

$$\forall \lambda \in \mathbb{R}^N_{\geq 0}, \quad \ln \mathbb{P}\left[\langle \lambda, \xi - \mathbb{E}[\xi]\rangle \geq d\right] \leq \inf_{t \geq 0} \left\{ -td + \sum_{k=1}^{N} \ln\left(\frac{\gamma_k e^{t\lambda_k b_k} + e^{-t\lambda_k b_k \gamma_k}}{1 + \gamma_k} \right) \right\} . \tag{21}$$

[1] A. Nemirovski and A. Shapiro. "Convex Approximations of Chance Constrained Programs". In: *SIAM Journal on Optimization* 17.4 (Jan. 2007), pp. 969–996

Distributionally robust knapsack problem

$$\max_{y\in\{0,1\}^N} \pi^T y \quad \text{s.t} \quad \sup_{F\in\mathcal{D}(\mu,\sigma,b)} \mathbb{P}_F\left[\xi^T y \geq c\right] \leq \tau$$

with uncertainty set

$$\mathcal{D}(\mu,\sigma,b) = \left\{ F \;\middle|\; \begin{array}{l} \mathbb{P}_F\left[|\xi_i-\mu_i| \leq b_i\right] = 1, \} \\ \mathbb{E}_F[\xi_i] = \mu_i,\;\; i = \{1,\ldots,N\} \\ \text{Var}(\xi_i) \leq \sigma_i^2 \end{array} \right\} .$$

Our approach:

$$\max_{\substack{y\in\{0,1\}^N \\ z\geq 0}} \pi^T y \quad \text{s.t} \quad \sum_{k=1}^{N} z \ln\left(\frac{\gamma_k e^{\frac{y_k}{z} b_k} + e^{-\frac{y_k}{z} b_k \gamma_k}}{1+\gamma_k}\right) - z\ln(\tau) + \mu^T y \leq c$$

Comparison with:

- Hoeffding: $\max_{y\in\{0,1\}^N} \pi^T y \quad \text{s.t} \quad \sqrt{2\ln(1/\tau)}\sqrt{y^T B y} + \mu^T y \leq c$
- Chebyshev-Cantelli: $\max_{y\in\{0,1\}^N} \pi^T y \quad \text{s.t} \quad \sqrt{\frac{1}{\tau}-1}\sqrt{y^T \Sigma y} + \mu^T y \leq c$

Entropic bounds

We define the ℓ_q-normof a vector $x \in \mathbb{R}^n$, $p \geq 1$, as:

$$\|x\|_q = \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} \quad .$$

We remind the known lower bounds of $\|x\|_0$ as ratios of norms ($\forall x \in \mathbb{R}^n \setminus \{0\}$):
We introduce a family of bounds generalizing the two previous bounds: for $x \neq 0$, and $\alpha > 0$, define

$$B_\alpha(x) := \left(\frac{\|x\|_1}{\|x\|_\alpha} \right)^{\frac{\alpha}{\alpha-1}} = \exp H_\alpha(p(x)) = \left(\sum_{i \in [n]} p_i(x)^\alpha \right)^{\frac{1}{\alpha-1}} \quad , \quad p(x) := |x| / \|x\|_1 .$$

In particular, B_1 simplifies to the exponential of the Shannon entropy.

$$B_1(x) = \frac{\|x\|_1}{\prod_{i \in [n]} |x_i|^{|x_i|/\|x\|_1}} = \|x\|_1 \exp \left(-\frac{1}{\|x\|_1} \sum_{i \in [n]} |x|_i \log |x|_i \right) \quad . \tag{22}$$

Monotonicity according to order α, see e.g. [Cac97]

$$B_\infty(x) \leq \cdots \leq B_2 \leq \cdots \leq B_1 \leq \cdots \leq B_0 = \|x\|_0 \quad . \tag{23}$$

Metric estimates between B_α and ϵ-cardinality

Let $\mathcal{A} \subset \mathbb{R}_+^n$. A real-valued function $\phi : \mathbb{R}_+^n \to \mathbb{R}$ is said to be *Schur-convex* (resp. *Schur-concave*) if $\phi(x) \leq \phi(y)$ (resp. $\phi(x) \geq \phi(y)$ for any $x, y \in \mathcal{A}$ satisfying $x \prec y$.

Proposition, see [MOA11], Appendix F.3.a (p.532)

The Rényi entropy of an arbitrary $\alpha > 0$ is Schur-concave.

We define the ϵ-cardinality as

$$\mathrm{card}_\epsilon(p) = |\{i \in [n] \mid p_i \geq \epsilon\}| \quad . \tag{24}$$

For any $\epsilon > 0$ and $0 < \alpha \leq 1$, an optimal solution of the problem

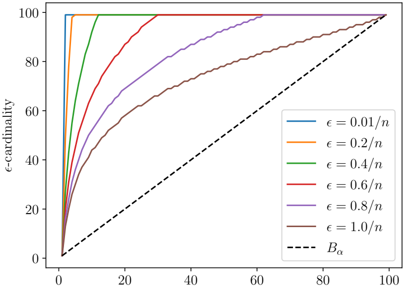
$$\min_{p \in \Delta_n} \{ H_\alpha(p) \mid \mathrm{card}_\epsilon(p) = k \} \tag{$P_{\alpha,\epsilon}^{k,n}$}$$

is $v_n(k, \epsilon)$, defined as

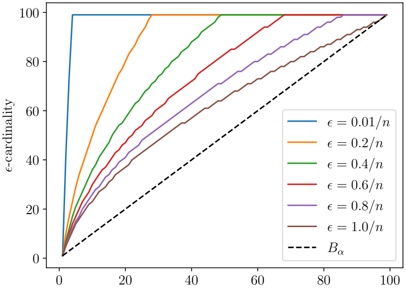
$$[v_n(k,\epsilon)]_i = \begin{cases} 1 - (k-1)\epsilon, & i = 1 \\ \epsilon, & 2 \leq i \leq k \\ 0, & k+1 \leq i \leq n \end{cases} \tag{25}$$

and corresponds to an objective value $\phi_{\alpha,\epsilon}(k)$.
As a conclusion, $\mathrm{card}_\epsilon(p) = k \Rightarrow B_\alpha(p) \geq \phi_{\alpha,\epsilon}(k)$, implying that $B_\infty(p) \leq b \Rightarrow \mathrm{card}_\epsilon(p) \leq \phi_{\alpha,\epsilon}^{-1}(b)$.

Metric estimates: numerical simulation



(a) $\alpha = 1$



(b) $\alpha = 0.5$